Modelling Financial Risks
Fat Tails, Volatility Clustering and Copulae

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Contents

Introduction

Overview

- Financial crisis has turned spotlight on risk management.
- Seconded by stricter regulatory framework.
- In this talk:
  - no debate whether quants have failed or not,
  - but some more recent techniques shall be outlined and elucidated with examples using R.

Introduction

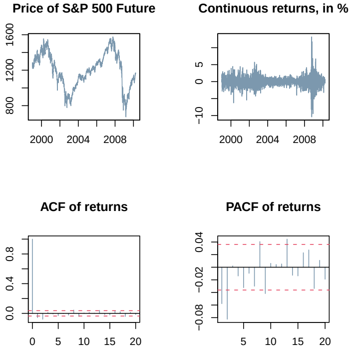
Stylized facts for single return series

- Daily returns though only marginally autocorrelated are usually not i.i.d.
- Volatility does not remain constant over time.
- Absolute or squared returns are strongly autocorrelated.
- Density of a return process is leptokurtic (*i.e.* fat tails).
- Clustering of extreme returns (*i.e.* volatility clustering).

Introduction

Example I: S&P 500 Future

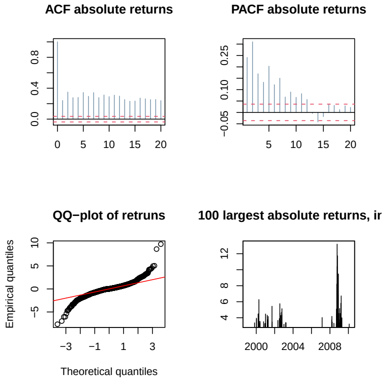
Figure: S&P 500 Future – stylized facts (I)



Introduction

Example II: S&P 500 Future

Figure: S&P 500 Future – stylized facts (II)



Introduction

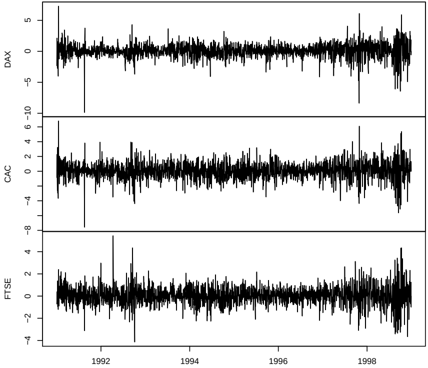
Stylized facts for multiple return series

- Simultaneous returns are significantly correlated, whereas cross-correlations are less pronounced.
- Absolute and squared returns exhibit clear correlation.
- Correlations of concurrent returns vary over time.
- Extreme values in a return series often correspond to extreme values in other time series.

Introduction

Example III: European Equity Markets

Figure: Continuous daily returns – stylized facts (III)



Introduction

Example IV: European Equity Markets

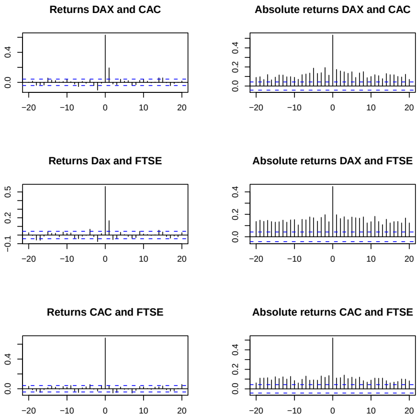
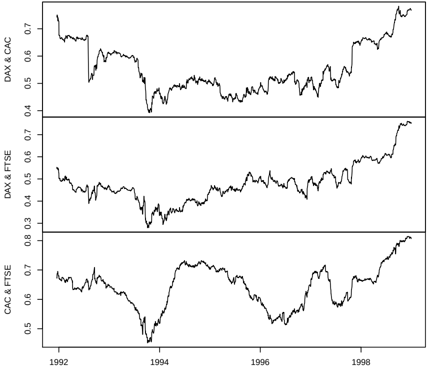


Figure: Cross-correlations of returns – stylized facts (IV)

Introduction

Example V: European Equity Markets

Figure: Return correlations (250 day moving window) – stylized facts (V)



Introduction

Losses as random variables

- Quantitative risk measures are based on a probability model.
- Wealth, V_t, is a random variable and is functionally related to time, t, and risk factors, Z_t.
- Future wealth, $V_{t+\Delta}$, is unknown and hence the loss: $L_{t,t+\Delta} = -(V_{t+\Delta} - V_t)$.
- As such the losses are random variables with a probability distribution, called the loss distribution (either conditional or unconditional if time-independent).

Introduction

Resources in R

Packages for Longitudinal Data:

- timeSeries
- xts
- zoo

Packages for Descriptive Data Analysis:

- fBasics
- fSeries
- fUtilities
- stats

Risk Measures

Value-at-Risk versus Expected Shortfall

Definition of VaR:

$$\begin{aligned} \text{VaR}_\alpha &= \inf\{l \in \Re : P(L > l) \leq 1 - \alpha\} \\ &= \inf\{l \in \Re : F_L(l) \geq \alpha\} \end{aligned} \tag{1}$$

Definition of modified VaR (Cornish-Fisher):

$$\begin{aligned} \text{mVaR}_\alpha =& \text{VaR}_\alpha + \frac{(q_\alpha^2 - 1)S}{6} + \\ & \frac{(q_\alpha^3 - 3q_\alpha)K}{24} - \frac{(2q_\alpha^3 - 5q_\alpha)S^2}{36} \end{aligned} \tag{2}$$

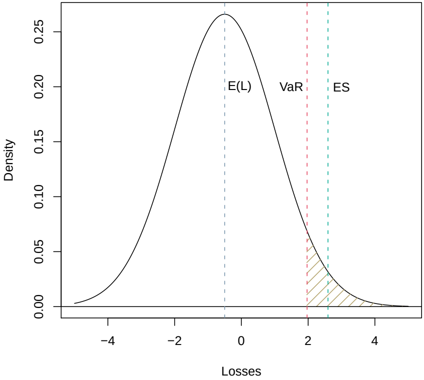
Definition of ES:

$$\begin{aligned} \text{ES}_\alpha &= \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_L)du \\ &= \frac{1}{1-\alpha}\int_\alpha^1 \text{VaR}_u(L)du \end{aligned} \tag{3}$$

Risk Measures

Graphical Display

Figure: Density function of the losses and risk measures



Risk Measures

Resources in R

Packages for Risk Measures:

- actuar
- fPortfolio
- PerformanceAnalytics
- QRMlib
- VaR

Nota bene: The risk measures are defined and calculated sometimes for the left- and not the right tail of the loss distribution.

Extreme Value Theory

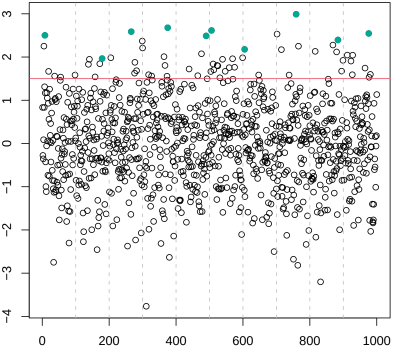
Block-Maxima versus Peaks-over-Threshold

- Basically, two procedures for extreme value modeling: block-maxima and peaks-over-threshold.
- Threshold, u, selection with Mean-Residual-Life plot.
- Distributions/Processes:
  - Generalized Extreme Value Distribution
  - Generalized Pareto Distribution
  - Poisson-Point-Process

Extreme Value Theory

Graphic: Block-Maxima versus PoT

Figure: Block-Maxima and Peaks-over-Threshold



Extreme Value Theory

PoT with GPD: Risk Measures

Distribution function of GPD:

$$H(y) = 1 - \left(1 + \frac{\xi y}{\tilde{\sigma}}\right)^{-1/\xi} \quad (4)$$

with $\tilde{\sigma} = \sigma + \xi(u - \mu)$ and $y : y > 0$.

VaR for GPD:

$$\mathrm{VaR}_\alpha = q_\alpha(F) = u + \frac{\tilde{\sigma}}{\xi}\left(\left(\frac{1-\alpha}{\bar{F}(u)}\right)^{-\xi} - 1\right) \quad (5)$$

ES for GPD:

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_x(F)dx = \frac{VaR_\alpha}{1-\xi} + \frac{\tilde{\sigma} - \xi u}{1-\xi} \quad (6)$$

Extreme Value Theory

GPD vs. Normal: Risk Simulation

- Daily returns of the S&P 500 Future
- Sample from 01/05/1999 to 06/02/2008
- Moving window of 1,000 observations
- Comparison of risk measure with the return of the next day.
- Hence, simulation starts at 11/05/2002 with a 1,455 data pairs
- Risk measure: ES with 99% level imply roughly 7 violations to be expected.
- For simplicity, count of data points for GPD kept fixed at twenty largest observations

Extreme Value Theory

GPD vs. Normal: Simulation Results I

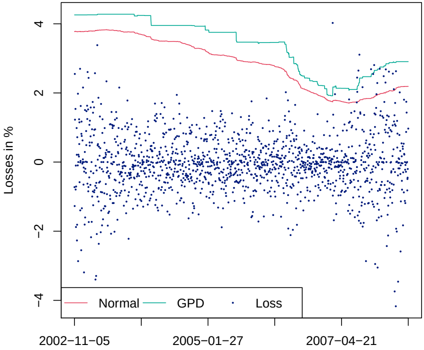
Table: Qualitative and quantitative results for ES

Model	Violation	Mean Error	Maximum Error
Normal	21	0.61	2.28
GPD	7	0.59	2.11

Extreme Value Theory

GPD vs. Normal: Simulation Results II

Figure: Losses and progression of ES



Extreme Value Theory

Resources in R

Packages for Extreme Value Theory:

- fExtremes
- ismev
- POT
- QRMlib

Distributions for Financial Returns

Introduction

- Concluded from stylized facts: Need for distributions that capture fat tails and asymmetries.
- Class of Generalized Hyperbolic Distrubtions (GHD)
- Commonly encountered sub-classes:
  - Hyperbolic distribution (HYP)
  - Normal Inverse Gaußian (NIG)

Distributions for Financial Returns

Generalized Hyperbolic Distribution (GHD)

Density:

$$\begin{aligned} \mathrm{gh}(x;\lambda,\alpha,\beta,\delta,\mu) =& a(\lambda,\alpha,\beta,\delta)(\delta^2+(x-\mu)^2)^{(\lambda-\frac{1}{2})/2} \\ & \times K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})\exp(\beta(x-\mu)) \ , \end{aligned} \tag{7}$$

with $a(\lambda,\alpha,\beta,\delta)$ defined as:

$$a(\lambda,\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda}K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})} \ , \tag{8}$$

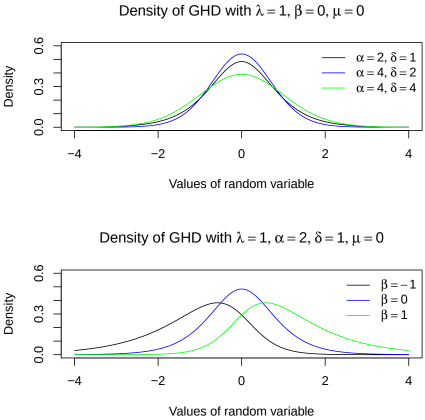
Often, GHD is in (ζ,ξ) notated (no location and scale):

$$\begin{aligned} \zeta &= \delta\sqrt{\alpha^2-\beta^2}, \ \rho = \beta/\alpha \\ \xi &= (1+\zeta)^{-1/2}, \ \chi = \xi/\rho \\ \bar{\alpha} &= \alpha\delta, \ \bar{\beta} = \beta\delta \ . \end{aligned} \tag{9}$$

Distributions for Financial Returns

Generalized Hyperbolic Distribution (GHD)

Figure: Densities of GHD-class



Distributions for Financial Returns

Hyperbolic Distribution (HYP)

The HYP is derived from GHD if $\lambda = 1$.

Density:

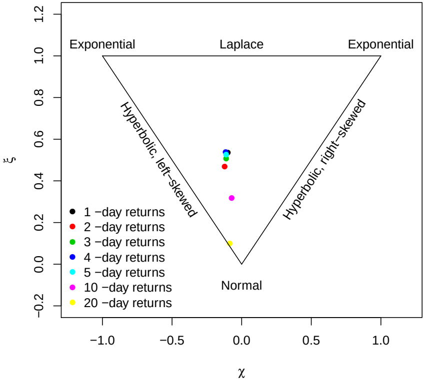
$$\text{hyp}(x; \alpha, \beta, \delta, \mu) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\delta\alpha K_1(\delta\sqrt{\alpha^2 - \beta^2}} \times \exp(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)) \quad (10)$$

with $x, \mu \in \mathfrak{R}$, $0 \leq \delta$ and $|\beta| < \alpha$. In (ξ, χ) notation the triangle relation $0 \leq |\chi| < \xi < 1$ holds (form triangle).

Distributions for Financial Returns

HYP: Form triangle for Eurex-Bund Future Returns

Figure: Form triangle with fitted HYP-parameters



Distributions for Financial Returns

Normal Inverse Gaußian Distribution (NIG)

The NIG is derived from GHD if $\lambda = \frac{-1}{2}$.

Density:

$$\begin{aligned}\text{nig}(x; \alpha, \beta, \delta, \mu) =& \frac{\alpha\delta}{\pi} \exp(\delta\sqrt{\alpha^2 - \beta^2} \\ &+ \beta(x - \mu)) \frac{K_1(\alpha\sqrt{\delta^2 + (x - \mu)^2})}{\sqrt{\delta^2 + (x - \mu)^2}}\end{aligned} \tag{11}$$

with parameter ranges: $x, \mu \in \mathfrak{R}$, $0 \leq \delta$ and $0 \leq |\beta| \leq \alpha$.

Distributions for Financial Returns

Resources in R

Packages for Generalized Hyperbolic Distribution:

- actuar
- fBasics
- ghyp
- HyperbolicDist
- QRMlib
- Runuran
- SkewHyperbolic

Conditional Volatility Modeling

Introduction

- Losses are now no longer assumed to be i.i.d.
- GARCH-model class are suited for capturing fat tails and volatility clustering (see stylized facts above).
- Volatility can directly be forecasted; no need for square-root-of-time rule, for instance.

Conditional Volatility Modeling

GARCH: Example I

- ESCB reference rate JPY/EUR (log-returns) from December 21, 1999 until October 10, 2008.
- Moving window of 250 obeservations.
- VaR for the 95% and 99% confidence level.
- Models: Normal distribution versus GARCH(1, 1) with Student's t innovations.
- Comparison of risk measure with next day's returns.

Conditional Volatility Modeling

GARCH: Example II

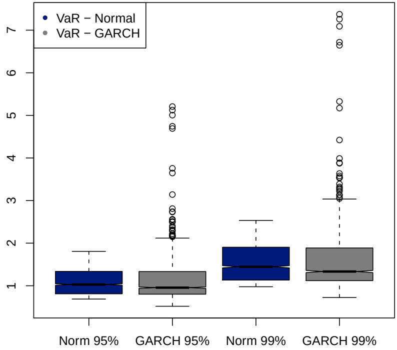
Table: VaR Results

Statistic	**VaR 95%**		**VaR 99%**	
	Normal	GARCH	Normal	GARCH
minimum	0.687	0.518	0.977	0.724
1st quantile	0.811	0.802	1.134	1.119
median	1.028	0.955	1.446	1.334
average	1.104	1.099	1.557	1.545
3rd quantile	1.336	1.333	1.905	1.887
maximum	1.807	5.207	2.534	7.375

Conditional Volatility Modeling

GARCH: Example III

Figure: Box Plots of VaR



Conditional Volatility Modeling

Resources in R

- bayesGARCH
- ccgarch
- fGarch
- gogarch
- rgarch (R-Forge)
- tseries

Modeling Dependence

Overview

- Copulae are a concept to model dependence between random variables.
- Copulae are distribution functions.
- Copulae concept: Bottom-up approach to multivariate model-building.
- Applications: Measure dependence, tail dependence, Monte Carlo studies.

Modeling Dependence

Definition

In prose: A d-dimensional copula is a distribution function on $[0,1]^d$ with standard uniform marginal distributions. Hence, the copula C is a mapping of the form $C : [0,1]^d \rightarrow [0,1]$, *i.e.*, a mapping of the unit hyper cube into the unit interval.

Modeling Dependence

Sklar's Theorem

Let F be a joint distribution function with margins $F_1, \dots, F_d$. Then there exists a copula $C : [0,1]^d \rightarrow [0,1]$ such that for all $x_1, \dots, x_d$ in $\bar{\mathbb{R}} = [-\infty, \infty]$,

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)) \quad . \tag{12}$$

If the margins are continuous, then C is unique; otherwise C is uniquely determined on $RanF_1 \times RanF_2 \times \cdots \times RanF_d$, where $RanF_i = F_i(\bar{\mathbb{R}})$ denotes the range of F_i. Conversely, if C is a copula and $F_1, \dots, F_d$ are univariate distribution functions, then the function F defined in (12) is a joint distribution function with margins $F_1, \dots, F_d$.

Modeling Dependence

Fréchet-Hoeffding bounds

If C is any d-copula, then for every $\mathbf{u}$ in $[0,1]^d$,

$$W^d(\mathbf{u}) \leq C(\mathbf{u}) \leq M^d(\mathbf{u}) \quad , \tag{13}$$

whereby

$$W^d(\mathbf{u}) = \max(\sum_{i=1}^{d} u_i + 1 - d, 0) \tag{14}$$

$$M^d(\mathbf{u}) = \min(u_1, \ldots, u_d) \tag{15}$$

The function $M^d(\mathbf{u})$ is a d-copula for $d \geq 2$, whereas the function $W^d(\mathbf{u})$ is not a copula for any $d \geq 3$. Please note, that these bounds hold for any multivariate df F.

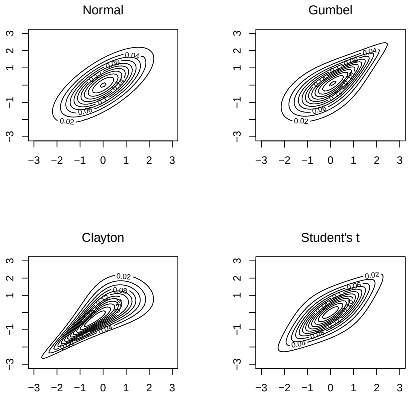
Modeling Dependence

Categories of copulas

- *Fundamental copulas*: These copulae represent important special dependence structures. Examples are: the independence copula, the comonotonicity copula (Fréchet-Hoeffding upper bound, perfectly positively dependent), the countermonotonicity copula (Fréchet-Hoeffding lower bound, perfectly negatively dependent)
- *Implicit copulas*: These copulae are extracted from well-known multivariate distributions using Sklar's Theorem. Ordinarily, these copulae do not possess simple closed-form expressions. Examples are: Gauß copula, t copula.
- *Explicit copulas*: These copulae have simple closed-form expressions. Examples are: Gumbel copula, Clayton copula.

Modeling Dependence

Copula: Example



Modeling Dependence

Confusion about Correlations

- "Among nine big economies, stock market correlations have averaged around 0.5 since the 1960s. In other words, for every 1% rise (or fall) in, say, American share prices, share prices in the other markets will typically rise (fall) by 0.5%." (The Economist, 8th November 1997)
- "A correlation of 0.5 does not indicate that a return from stock-market A will be 50% of stockmarket B's return, or vice-versa ... A correlation of 0.5 shows that 50% of the time the return of stockmarket A will be positively correlated with the return of stock market B, and 50% of the time it will not." (The Economist (letter), 22nd November 1997)

Modeling Dependence

Correlation pitfalls I

- The use of correlation coefficients as a measure of dependence and risk allocation between risky assets is widespread.
- However, applying correlation coefficients blindly to multivariate data sets might be misleading.
- Working with correlation coefficients is unproblematic in the case of jointly normally distributed series. (this holds true for all elliptical distributions).

Modeling Dependence

Correlation pitfalls II

Fallacy 1: Marginal distributions and correlation determine the joint distribution.

- Only true, if assets are following an elliptical distribution.
- If the series are non-elliptically distributed, then there are infinitely many distributions that will fit the data.
- Correlation coefficients do not contain information about tail-dependencies between risky assets.

Modeling Dependence

Correlation pitfalls III

Fallacy 2: Given marginal distributions F_1 and F_2 for X_1 and X_2, all linear correlations between -1 and 1 can be attained through suitable specification of the joint distribution F.

- Only true, if assets are following an elliptical distribution.
- In general, the attainable correlations depend on F_1 and F_2 and form a closed interval $[\rho_{min}, \rho_{max}]$ containing zero that is a subset of $[-1, 1]$.
- For instance, given a bivariate log-normal distribution the valid range of ρ is $[-0.090, 0.666]$.
- Hence, a low correlation does not point to a low dependence between two random variables!

Modeling Dependence

Correlation pitfalls: Summary

1. Correlation is simply a scalar measure of dependency; it cannot tell us everything we would like to know about the dependence structure of risks.
2. Possible values of correlation depend on the marginal distribution of the risks. All values between -1 and 1 are not necessarily attainable.
3. Perfectly positively dependent risks do not necessarily have a correlation of 1; perfectly negatively dependent risks do not necessarily have a correlation of -1.
4. A correlation of zero does not indicate independence of risks.
5. Correlation is not invariant under transformations of the risks. For example, $\log(X)$ and $\log(Y)$ generally do not have the same correlation as X and Y.
6. Correlation is only defined when the variances of the risks are finite. It is not an appropriate dependence measure for very heavy-tailed risks where variances appear infinite.

Modeling Dependence

Fitting Copulas to data

- Methods-of-Moments using Rank Correlation (Spearman and Kendall)
- Forming Pseudo-sample from the copula (parametric and non-parametric estimation and/or EVT for the tails).
- Maximum-Likelihood Estimation.

Modeling Dependence

Rank correlation coefficients

- Spearman's rank correlation coefficient:

$$\frac{12}{n(n^2-1)}\sum_{t=1}^{n}(rank(X_{t,i})-\frac{1}{2}(n+1))(rankX_{t,j})-\frac{1}{2}(n+1))$$

- Kendall's tau:

$$\binom{n}{2}^{-1}\sum_{1\leq t<s\leq n} sign((X_{t,i}-X_{s,i})(X_{t,j}-X_{s,j}))$$

Modeling Dependence

Coefficients of Tail Dependence I

- Coefficients of tail dependence are measures of pairwise dependence that depend only on the copula of a pair of rvs X_1 and X_2.
- These coefficients provide a measure of extremal dependence, *i.e.*, the dependence in tails of the distribution.
- Here, the measures are defined in terms of limiting conditional probabilities of quantile exceedances.

Modeling Dependence

Coefficients of Tail Dependence II

Definition

Let X_1 and X_2 be rvs with dfs F_1 and F_2. The coefficient of upper dependence of X_1 and X_2 is:

$$\lambda_u := \lambda_u(X_1, X_2) = \lim_{q \to 1^-} P(X_2 > F_2^{-1}(q) | X_1 > F_1^{-1}(q)) \quad ,$$

provided a limit $\lambda_u \in [0, 1]$ exists. If $\lambda_u \in [0, 1]$, then X_1 and X_2 are said to show upper tail dependence or extremal dependence in the upper tail; if $\lambda_u = 0$, they are asymptotically independent in the upper tail. Analogously, the coefficient of lower tail dependence is:

$$\lambda_l := \lambda_l(X_1, X_2) = \lim_{q \to 0^+} P(X_2 \leq F_2^{-1}(q) | X_1 \leq F_1^{-1}(q)) \quad ,$$

provided a limit $\lambda_l \in [0, 1]$ exists.

Modeling Dependence

Coefficients of Tail Dependence III

- Upper tail dependence for the Gumbel copula: $\lambda_u = 2 - 2^{1/\theta}$ for $\theta > 1$.
- Lower tail dependence for the Clayton copula: $\lambda_l = 2^{-1/\theta}$ for $\theta > 0$.
- Because of its symmetry the lower and upper tail dependence coefficients are equal for the Gauß and t copulae. It can be shown that the Gauß copula is asymptotically independent in both tails. For the t copula the coefficient of tail dependence is defined as:

$$\lambda = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right) \quad ,$$

provided that $\rho > -1$.

Copula-GARCH

Introduction

- Combination of GARCH-models for the marginal distributions and capturing the dependencies between these with a copula.
- GARCH specifications can be different for risk factors.
- Fat tails and/or asymmetries are explicitly taken into account.
- Risk measures are calculated by Monte-Carlo simulations.

Copula-GARCH

Step-by-step guide

1. Specify and estimate GARCH models.
2. Retrieve standardised residuals.
3. Convert to pseudo-uniform variables (either according to the distribution assumption or empirically).
4. Estimate copula.
5. Simulate N data sets from copula by Monte-Carlo.
6. Calculate the quantiles and the simulated losses.
7. Obtain the desired risk measure.

Copula-GARCH

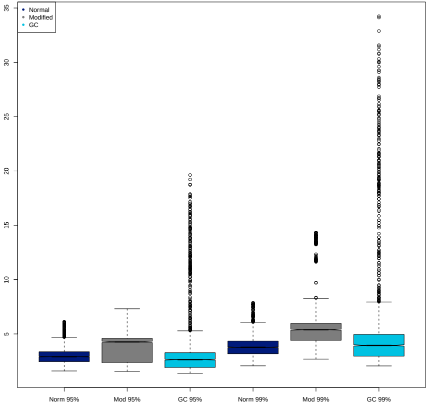
Copula-GARCH: Simulation

- Equally weighted portfolio of five US companies: Bank of America, Citigroup, General Motors, Procter & Gamble and United Technologies.
- Sample period from 30 December 1994 to 30 April 2009
- Rolling window of 1,000 obersvations, hence simulation starts at the 3rd November 1998 and contains 2738 data sets.
- Comparison of ES with the subsequent portfolio return.
- Models: Normaldistribution vs. GARCH(1, 1) with Student's t innovations and a Student's t copula.

Copula-GARCH

Copula-GARCH: Box Plots

Figure: Box Plots of ES



Copula-GARCH

Copula-GARCH: Time Series Plots

Figure: Losses and ES 95%

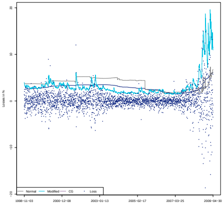
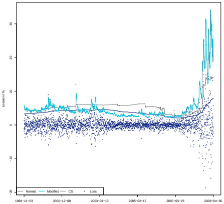


Figure: Losses and ES 99%



Copula-GARCH

Copula-GARCH: Cumulated non-anticipated losses

Figure: Draw Downs 95%

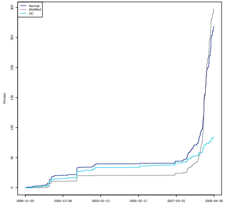
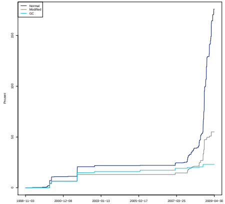


Figure: Draw Downs 99%



Summary

- Many packages are already available that are focused on risk modelling.
- Unfortunately, risk measures are not defined consistently.
- Lack of out-of-the-box methods for the more elaborated risk models.

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